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The Convergence of Expansion Method of Chebyshev Polynomials

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Abstract

In this paper, the weakly singular linear and nonlinear integro-differential equations are solved by using expansion method of Chebyshev polynomials of the first kind .The approximation solution of this equation is calculated in the form of a series which its components are computed easily .The existence and uniqueness of the solution and the convergence of the proposed method are proved. Numerical examples are studied to demonstrate the accuracy of the presented method.

Keywords: China insurance industry, Volterra integral equations, Fredholm integral equations, Integro-differential equations, Singular integral equations, Chebyshev polynomials method.

1. Introduction

Integro - differential equations play an importance role in scientific field such as fluid dynamics, solid state physics plasma physics and mathematical biology (Belbas, 2007). Also, many researchers have been studied convergence analysis of integral equations, weakly singular integral equations and weakly singular integro-differential equations say. In 1992, Tang, (Tang, 1992) founded super convergence of numerical solutions to Volterra weakly singular integro-differential equations.

In 2007, Maleknejad and et al, (Maleknejad, 2007) discussion on convergence of Legendre polynomial for numerical solution of integral equations. In 2008, Kangro and et al, (Kangro, 2008) founded convergence of spline collocation for Volterra integral equations.

In 2012, Sadigh and Yildirim, (Sadigh, 2012) used a method to estimate the solution of a non linear weakly singular integro-differential equations by applying the homotopy methods. In 2014, Jafarian and et al, (Jafarian, 2014) founded convergence analysis of Bernstein polynomials method to the system of weakly singular integral equations.

This paper is organized as follows. In section two definitions of Chebyshev polynomials of the first kind and its properties. In section three existence and uniqueness solution of linear Fredholm weakly singular integro-differential equations (LFWSIDEs), nonlinear Fredholm weakly singular integro-differential equations (NLFWSIDEs), linear Volterra weakly singular integro-differential equations (LVWSIDEs), Nonlinear Volterra Weakly singular integro-differential equations (NLVWSIDEs). In section four convergence analysis of the proposed method. Applications are presented in section five. Finally, a brief discussion is stated in last section.

2. Chebyshev polynomials of the first kind $\mathbf{T_n(x)}$, (Mason, 2003)

The Chebyshev polynomials of the first kind of degree n is a set of orthogonal polynomials and it is defined by the recurrence relation

$T_0(x) = 1$

$T_1(x) = x$

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \qquad \text{for each } n \geq 1. \qquad \ldots(1)$$

2.1 Properties of the Chebyshev polynomials $T_n(x)$

The following important properties about Chebyshev polynomials of the first kind $T_n(x)$ of degree n.

1. The Chebyshev polynomials of the first kind $T_n(x), n = 0,1, \ldots$ are

a set of orthogonal polynomials over the interval [-1,1] with respect to the weight function $w(x) = (1 - x^2)^{-1/2}$, that is :

$$\int_{-1}^{1} w(x)T_n(x)T_m(x)dx = \begin{cases} 0 & n \neq m \\ \frac{\pi}{2} & n = m \neq 0 \\ \pi & n = m = 0 \end{cases} \quad \ldots(2)$$

2. The Chebyshev polynomials of the first kind can be defined by the trigonometric identity $T_n(\cos(\theta)) = \cos(n\theta)$ for $n = 0,1,2,3,\ldots$
3. $T_n(x)$ has n distinct real roots x_i on the interval [-1,1], these roots are defined by :

$$x_i = \cos\left(\frac{(2i+1)\pi}{2N}\right), \quad i = 0,1,2,\ldots,N-1 \quad \ldots(3)$$

are called Chebyshev nodes.$T_n(x)$ assumes its absolute extrema at

$$x_j = \cos\left(\frac{j\pi}{N}\right) \quad \text{for } j = 0,1,2,\ldots,N \quad \ldots(4)$$

4. A polynomial of degree N in Chebyshev form is a polynomial

$$p(x) = \sum_{n=0}^{N} a_n T_n(x) \quad \ldots(5)$$

Where T_n is the n^{th}Chebyshev form

The first few Chebyshev polynomials of the first kind for N=0,1,2,3,4,5 are given in figure (1).

2.2 Shifted Chebyshev Polynomials

Shifted Chebyshev polynomial are also of interest when the range of the independent variable is [0,1] instead of [-1,1].The Shifted Chebyshev polynomials of the first kind are defined as

$$T_n^*(x) = T_n(2x-1), \quad 0 \leq x \leq 1 \quad \ldots(6)$$

Similarly, one can also build Shifted polynomials for a generic interval [a,b].Where

$$\bar{x}_i = \frac{b-a}{2}x_i + \frac{b+a}{2} \quad \ldots(7)$$

The first few Chebyshev polynomials of the first kind for N=0,1,2,3,4,5 for interval [0,1] are given in figure (2)

3. Existence and Uniqueness Solution of NLVWSIDEs

We consider the following k^{th}-order NLVWSIDEs of the following form:

$$\sum_{j=0}^{k} p_j(x)y^{(j)}(x) = f(x) + \int_a^x k(x,t,y(t))dt, \quad x \in [a,b] \quad \ldots \quad (8)$$

with initial conditions

$$y^{(r)}(a) = \beta_r, r = 0,1,\ldots,k-1$$

Where a, b, β_r are constant values, $k(x,t,y(t)) = \frac{H(x,t,y(t))}{|x-t|^{1/2}}$, $f(x)$, $H(x,t,y(t))$ are functions which have suitable derivatives on an interval

$a \leq t \leq x \leq b$ and $p_j(x)$, $j = 0,1,\ldots,k$ that $p_k(x) \neq 0$ are given , and

$y(x)$ is the solution to be determined .To solve Eq.(9), we consider (9) as follows:

$$y(x) = L^{-1}\left(\frac{f(x)}{p_k(x)}\right) + \sum_{r=0}^{k-1}\frac{1}{(r!)}(x-a)^r b_r + L^{-1}\left(\int_a^x \frac{H(x,t,y(t))}{p_k(t)|x-t|^{1/2}}dt\right) - L^{-1}\left(\sum_{j=0}^{k-1}\frac{p_j(x)}{p_k(x)}y^{(j)}(x)\right). \quad \ldots \quad (9)$$

Where L^{-1} is the multiple integration operator as follows:

$$L^{-1}(.) = \int_a^x \int_a^x \ldots \int_a^x \int_a^x (.)dxdx\ldots dxdx. \quad (k \text{ times})$$

We can obtain the term $\sum_{r=0}^{k-1}\frac{1}{(r!)}(x-a)^r b_r$ from the initial conditions. the following relations have been mentioned in [50], we have

$$L^{-1}\left(\int_a^x \frac{H(x,t,y(t))}{p_k(t)|x-t|^{1/2}}dt\right) = \int_a^x \frac{(x-t)^k}{(k!)}\frac{H(x,t,y(t))}{p_k(t)|x-t|^{1/2}}dt, \quad \ldots(10)$$

$$\sum_{j=0}^{k-1} L^{-1}\left(\frac{p_j(x)}{p_k(x)}\right) y^{(j)}(x)) = \sum_{i=0}^{k-1}\int_a^x \frac{(x-t)^{k-1}}{(k-1)!}\frac{p_j(x)}{p_k(x)} y^{(j)}(x)dt, \qquad \ldots \quad (11)$$

By substituting (10) and (11) into (9), we obtain

$$y(x) = L^{-1}\left(\frac{f(x)}{p_k(x)}\right) + \sum_{r=0}^{k-1}\frac{1}{(r!)}(x-a)^r b_r + \int_a^x \frac{(x-t)^k}{(k!)}\frac{H(x,t,y(t))}{p_k(t)|x-t|^{1/2}}dt - \sum_{i=0}^{k-1}\int_a^x \frac{(x-t)^{k-1}}{(k-1)!}\frac{p_j(x)}{p_k(x)} y^{(j)}(x)dt..(12)$$

For convenient, we set

$$L^{-1}\left(\frac{f(x)}{p_k(x)}\right) + \sum_{r=0}^{k-1}\frac{1}{(r!)}(x-a)^r b_r = F(x)$$

$$\frac{(x-t)^k}{(k!)p_k(t)|x-t|^{1/2}} = g_1(x,t) \quad \frac{(x-t)^{k-1}}{(k-1)!}\frac{p_j(x)}{p_k(x)} = g_2(x,t) \qquad \ldots(13)$$

So, we have

$$y(x) = F(x) + \int_a^x g_1(x,t)k(x,t,y(t))dt - \sum_{i=0}^{k-1}\int_a^x g_2(x,t)y^{(i)}(t)dt. \qquad \ldots \quad (14)$$

Consider the Eq.(14), we assume $F(x)$ is bounded for all x and
$|g_1(x,y)| \le M_1$,
$|g_2(x,y)| \le M_{1_j}$, $j = 0,1,\ldots,k-1, \forall\, x \in [a,b]$.
Also, we suppose the nonlinear terms $k(x,t,y(t))$ and $D^j(y(x))$ are Lipschitz continuous with $|k(y(x) - k(y^*(x))| \le l|y(x) - y^*(x)|$,
$|D^j(y(x)) - D^j(y^*(x))| \le R_j|y(x) - y^*(x)|$, $j = 0,1,\ldots,k-1$.
If we set
$\gamma = (b-a)(lM_1 + kRM_2)$,
$R = \max|R_j|$, $M_2 = \max\left|M_{1_j}\right|, j = 0,1,\ldots,k-1$.
In what follow, we will prove theorem by considering the above assumptions.

Theorem (3.1) :

The NLVWSIDEs in Eq.(9), has a unique solution Whenever $0 < \gamma < 1$.

Proof.

Let y and y^* be two different solutions of (15) then

$|y(x) - y^*(x)| = \left|\int_a^x g_1(x,t)[k(x,t,y(t)) - k(x,t,y^*(t))]dt - \sum_{j=0}^{k-1}\int_a^x g_2(x,t)[D^j(y(t)) - D^j(y^*(t))]dt\right| \le \int_a^x |g_1(x,t)||k(x,t,y(t)) - k(x,t,y^*(t))|dt + \sum_{j=0}^{k-1}\int_a^x |g_2(x,t)||D^j(y(t)) - D^j(y^*(t))|dt \le (b-a)(lM_1 + kRM_2)|y(x) - y^*(x)| = \gamma|y(x) - y^*(x)|$

From which we get $(1-\gamma)|y(x) - y^*(x)| \le 0$. Since $0 < \gamma < 1$, so
$|y - y^*| = 0$. Therefore, $y = y^*$ and this completes the proof. □

4. Convergence Analysis

We proffer some theorems, as a background which will be needed to prove the convergence analysis of Chebyshev polynomials of the first kind.

Theorem(4.1), (OT, 2015):

The polynomial $\widehat{T}_n(x) = 2^{1-n}T_n(x)$ is the minimax approximation on $[-1,1]$ to the zero function by ammonic polynomial of degree n and

$$\|\widehat{T}_n\| = 2^{1-n}. \qquad \ldots(16)$$

Theorem (4.2), (OT, 2015):

Let $P_n \in \mathbb{P}_n$ be the Chebyshev interpolation polynomial for y at n+1 points .Then

$$\|y - P_n\| \leq M(n), \text{ with } M(n) \sim C\omega(1/n)\log n, \quad \ldots(17)$$

as $n \longrightarrow \infty, C$ being a constant.

Theorem (4.3) :

Let $y_n(x)$ be polynomials of degree n that the numerical coefficients are created by solving the nonlinear system (21). There exists an integer N such that, for all $n \geq N$, these polynomials converge to the exact solution of the integral equation (9)

Proof:

We know that the Chebyshev polynomials for the function $y(x)$ that is continuous on interval $[0,1]$ converge to this function (Th.4.1) and Th.4.2).Consider again (9). In other statement, $y(x)$ can be expanded as a uniformly convergent Chebyshev polynomials in $[0,1]$:

$$y(x) = \sum_{i=0}^{n} b_i T_i^*(x)\ , n \geq N \quad \ldots(18)$$

Based on the submitted operation, WSIDEs (9) can be transformed to the following equivalent infinitely systems of nonlinear equations for unknown $y(x)$:

$$AX = B \quad (19)$$

With

$$A = \lim_{n\to\infty} A_{n,n},\ B = \lim_{n\to\infty} B_n,\quad X = \lim_{n\to\infty} X_n, \quad \ldots(20)$$

where $A_{n,n}$, B_n, and X_n were defined as follows.

$$\begin{bmatrix} A_{0,0} & \cdots & A_{0,n} \\ \vdots & \ddots & \vdots \\ A_{n,0} & \cdots & A_{n,n} \end{bmatrix} \begin{bmatrix} X_0 \\ . \\ . \\ X_n \end{bmatrix} = \begin{bmatrix} B_0 \\ . \\ . \\ B_n \end{bmatrix}, \quad \ldots(21)$$

With $n \times (n+1)$ non linear equations of $n \times (n+1)$ unknowns coefficients .Or equivalently

$$A_{n,n} X_n = B_n, \quad \ldots(22)$$

With the block matrices

$$A_{i,j} = \begin{bmatrix} a_{i,j}^{0,0} & \cdots & a_{i,j}^{0,n} \\ \vdots & \ddots & \vdots \\ a_{i,j}^{n,0} & \cdots & a_{i,j}^{n,n} \end{bmatrix}, \quad X = \begin{bmatrix} x_{j,0} \\ . \\ . \\ x_{j,n} \end{bmatrix}, \ B = \begin{bmatrix} b_i(x_0) \\ . \\ . \\ b_i(x_n) \end{bmatrix}. \quad \ldots \quad (23)$$

For the above system ,the unique solution can be expressed as

$$X = A^{-1}B, \quad \ldots(24)$$

Alternatively, the above system can be rewritten as

$$\begin{bmatrix} X^n \\ X^\infty \end{bmatrix} = \begin{bmatrix} {A^{-1}}_{n,n} & {A^{-1}}_{n,\infty} \\ {A^{-1}}_{\infty,n} & {A^{-1}}_{\infty,\infty} \end{bmatrix} \begin{bmatrix} B_n \\ B_\infty \end{bmatrix}, \quad \ldots(25)$$

Where

$$(A^{-1})_{n,n} = \begin{bmatrix} (A^{-1})_{0,0}^{n,n} & (A^{-1})_{0,n}^{n,n} \\ (A^{-1})_{n,0}^{n,n} & (A^{-1})_{n,n}^{n,n} \end{bmatrix} \quad \ldots(26)$$

Consequently, one finds that the y_n composed of the first n+1 elements of the exact solution vector y must satisfy the following relation:

$$X_n = (A^{-1})_{n,n} B_n + (A^{-1})_{n,\infty} B_\infty \quad \ldots(27)$$

In addition, based on the analysis of the foregoing section, the unique solution of (5.11) is denoted as

$\overline{X}_n = (A_{n,n})^{-1}B_n$ …(28)

Subtracting (28) from (27) yields

$$X_n - \overline{X}_n = (A^{-1})_{n,n}B_n + (A^{-1})_{n,\infty}B_\infty - (A_{n,n})^{-1}B_n$$
$$= \left((A^{-1})_{n,n} - (A_{n,n})^{-1}\right)B_n + (A^{-1})_{n,\infty}B_\infty$$

$X_n - \overline{X}_n = E_{n,n}B_n + (A^{-1})_{n,\infty}B_\infty$ …(29)

Where $E_{n,n} = (A^{-1})_{n,n} - (A_{n,n})^{-1}$. Expanding the right-hand side of (29), the left- hand side of this is expressed by

$y(x) - y_n(x) = \sum_{i=0}^{n}\sum_{j=0}^{n} e_{i,j}f(x_i) + \sum_{i=0}^{n}\sum_{j=n+1}^{\infty} a^{-1}{}_{i,j}f(x_i)$ … (30)

Where $e_{i,j}$ and $a^{-1}{}_{i,j}$ are the elements of E and A^{-1} respectively .Thus,

$|y(x) - y_n(x)| \le ((\sum_{i=0}^{n}\sum_{j=0}^{n}\left|e_{i,j}\right|^2))^{1/2} - (\sum_{i=0}^{n}\sum_{j=n+1}^{\infty}|a^{-1}{}_{i,j}|^2)^{1/2}$,

$(\sum_{i=0}^{n}\sum_{j=0}^{n}|f(x_i)|^2)^{1/2}(\sum_{i=0}^{n}\sum_{j=0}^{n}|B(x)|^2)^{1/2}$ …(31)

follows from the well-known Cauchy–Schwarz inequality since $\lim_{n\to\infty} E_{n,n} = 0$ and $\lim_{n\to\infty} A^{-1}{}_{n,\infty} = 0$ so we can conclude that $\lim_{n\to\infty}|y(x) - y_n(x)| = 0$and proof is completed. □

Remark (1):

If $k\left(x,t,y(t)\right) = k(x,t)y(t)$ then the equation

$p(x)y'(x) - y(x) - \int_a^x k(x,t)y(t)dt = f(x)\ x \in [a,b]$

is a LVWSIDEs and the previous proof will be special case of NLVWSIDEs.

Remark (2):

If $k\left(x,t,y(t)\right) = k(x,t)y(t)$ then the equation

$\mu_1 y'(x) + \mu_2 y(x) + \int_a^b k(x,t)y(t)dt = f(x) \quad x \in [a,b]$

is a LFWSIDEs and the previous proof will be special case of NLVWSIDEs.

Remark (3):

If $k\left(x,t,y(t)\right) = k(x,t)y(t)$ then the equation

$y'(x) + p(x)y(x) + \int_a^b k\left(x,t,y(t)\right)dt = f(x) \quad x \in [a,b]$

is a NLFWSIDEs and the previous proof will be special case of NLVWSIDEs.

5. Applications

Application (1)

Consider the following LFWSIDEs, (Mehrdad, 2001):

$y'(x) + \int_0^1 |\tilde{x} - \tilde{t}|^{-1/2}y(\tilde{t})d\tilde{t} = f(\tilde{x})$ …(32)

with initial condition $y(0) = 1$

here the forcing function f is selected such that the exact solution is

$y(\tilde{x}) = \tilde{x}^2 + \frac{1}{\tilde{x}+1}$

Tables 1, 2 and 3 illustrate the comparison between the exact and the approximate solution depending on Mean Square Error (MSE) and Elapsed Time(ET) for N=10,12,14.

Figure 3, 4, 5 illustrate the comparison between the exact and the approximate solution of application (1) when N= 10, 12 and 14 respectively.

Figure 6 illustrate the comparison between the shifted Chebyshev nodes and the MSE of application (1) when N=2…14.

Application (2)

Consider the following first –order NLVWSIDEs, (Xueqin, 2012) :

$$y'(\tilde{x}) + p(\tilde{x})y(\tilde{x}) = f(\tilde{x}) + \int_a^{\tilde{x}} \frac{F(y(\tilde{t}))}{(\tilde{x}-\tilde{t})^\alpha} d\tilde{t}, 0 < \alpha < 1, \tilde{x} \in [0,1] \quad \ldots(33)$$

with initial condition $y(0) = 0$ where $\alpha = \frac{1}{2}, a = 0$, and the exact solution $y(\tilde{x}) = \tilde{x}(\tilde{x} - 1)$. Let $F(y(\tilde{t})) = y^2(\tilde{t})$,

$$p(\tilde{x}) = \left(\frac{16}{315}\right)\tilde{x}^{\frac{5}{2}}\big(21 + 4\tilde{x}(4\tilde{x} - 9)\big) + 1, f(\tilde{x}) = \tilde{x}^2 + \tilde{x} - 1.$$

Tables 4, 5 and 6 illustrate the comparison between the exact and the approximate solution for N=10,12 and 14 depending on Mean Square Error (MSE) and Elapsed Time (ET).

Figure 7, 8, 9 illustrate the comparison between the exact and the approximate solution of application (2) when N= 10, 12 and 14 respectively.

Figure 10 illustrate the comparison between the shifted Chebyshev nodes and the MSE of application (2) when N=2…14.

6. Conclusions

In this paper, we discussed the existence and uniqueness solution to the NLVWSIDEs, LVWSIDEs, LFWSIDEs, NLFWSIDEs and convergence analysis of the proposed method to the NLVWSIDEs. The results show that:

- The expansion method of Chebyshev polynomials of the first kind as basis function is actively, easily.
- The expansion method of Chebyshev polynomials of the first kind as basis function is accurately a large class of linear and nonlinear problems with the approximations which are convergent are speedily to the exact solutions.

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Table 1: Results obtained and errors for application (1): N=10.

x̃-values	**Exact solution**	**Approximate solution**	**Absolute error**
0.0051	1.000000	1.000001	0.000001
0.0452	0.919091	0.919092	0.000001
0.1221	0.873333	0.873334	0.000001
0.2297	0.859231	0.859231	0.000001
0.3591	0.874286	0.874286	0.000001
0.5	0.916667	0.916667	0.000000
0.6409	0.985000	0.985000	0.000000
0.7703	1.078235	1.078235	0.000000
0.8779	1.195556	1.195556	0.000000
0.9548	1.336316	1.336316	0.000000
0.9949	1.500000	1.500000	0.000000
MSE	**2.5217e-013**	**ET**	**63.371781Sec.**

Table 2: Results obtained and errors for application(1): N=12

x̃-values	**Exact solution**	**Approximate solution**	**Absolute error**
0.0037	1.000000	1.000000	0.000000
0.0325	0.930021	0.930021	0.000000
0.0885	0.884921	0.884921	0.000000
0.1684	0.862500	0.862500	0.000000
0.2676	0.861111	0.861111	0.000000
0.3803	0.879493	0.879493	0.000000
0.5	0.916667	0.916667	0.000000
0.6197	0.971857	0.971857	0.000000
0.7324	1.044444	1.044444	0.000000
0.8316	1.133929	1.133929	0.000000
0.9115	1.239899	1.239899	0.000000
0.9675	1.362017	1.362017	0.000000
0.9964	1.500000	1.500000	0.000000
MSE	**3.6854e-016**	**ET**	**216.990689Sec.**

Table 3: Results obtained and errors for application(1): N=14

x̃-values	**Exact solution**	**Approximate solution**	**Absolute error**
0.0027	1.000000	1.000000	0.000000
0.0245	0.938435	0.938435	0.000000
0.067	0.895408	0.895408	0.000000
0.1284	0.869448	0.869448	0.000000
0.2061	0.859410	0.859410	0.000000
0.2966	0.864393	0.864393	0.000000
0.3960	0.883673	0.883673	0.000000
0.5	0.916667	0.916667	0.000000
0.604	0.962894	0.962894	0.000000
0.7034	1.021961	1.021961	0.000000
0.7939	1.093537	1.093537	0.000000
0.8716	1.177347	1.177347	0.000000
0.933	1.273155	1.273155	0.000000
0.9755	1.380763	1.380763	0.000000
0.9973	1.500000	1.500000	0.000000
MSE	**3.4425e-018**	**ET**	**899.911312Sec.**

Table 4: Results obtained and errors for application (1): N=10.

x̃-values	**Exact solution**	**Approximate solution**	**Absolute error**
0.0051	-0.0051	-0.0234	0.0183
0.0452	-0.0431	-0.0593	0.0162
0.1221	-0.1072	-0.1193	0.0121
0.2297	-0.1769	-0.1828	0.0059
0.3591	-0.2302	-0.2268	0.0034
0.5	-0.2500	-0.2344	0.0156
0.6409	-0.2302	-0.2018	0.0283
0.7703	-0.1769	-0.1387	0.0382
0.8779	-0.1072	-0.0631	0.0441
0.9548	-0.0431	0.0038	0.0470
0.9949	-0.0051	0.0430	0.0481
MSE	**8.8719e-004**	**ET**	**164.313091Sec.**

Table 5: Results obtained and errors for application(1): N=12

$\tilde{x}$-values	Exact solution	Approximate solution	Absolute error
0.0037	-0.0036	-0.0164	0.0127
0.0325	-0.0314	-0.0429	0.0114
0.0885	-0.0807	-0.0896	0.0090
0.1684	-0.1401	-0.1453	0.0053
0.2676	-0.1960	-0.1959	0.0001
0.3803	-0.2357	-0.2279	0.0078
0.5	-0.2500	-0.2325	0.0175
0.6197	-0.2357	-0.2080	0.0277
0.7324	-0.1960	-0.1598	0.0362
0.8316	-0.1401	-0.0981	0.0420
0.9115	-0.0807	-0.0354	0.0453
0.9675	-0.0314	0.0155	0.0470
0.9964	-0.0036	0.0440	0.0477
MSE	**8.5709e-004**	**ET**	**420.942616Sec.**

Table 6: Results obtained and errors for application(1): N=14

$\tilde{x}$-values	Exact solution	Approximate solution	Absolute error
0.0027	-0.0027	-0.0120	0.0093
0.0245	-0.0239	-0.0323	0.0084
0.067	-0.0625	-0.0693	0.0068
0.1284	-0.1119	-0.1163	0.0043
0.2061	-0.1636	-0.1645	0.0009
0.2966	-0.2086	-0.2047	0.0039
0.396	-0.2392	-0.2287	0.0105
0.5	-0.2500	-0.2314	0.0186
0.604	-0.2392	-0.2121	0.0271
0.7034	-0.2086	-0.1741	0.0346
0.7939	-0.1636	-0.1234	0.0402
0.8716	-0.1119	-0.0682	0.0438
0.933	-0.0625	-0.0167	0.0458
0.9755	-0.0239	0.0230	0.0469
0.9973	-0.0027	0.0447	0.0474
MSE	**8.4706e-004**	**ET**	**5641.463816Sec.**

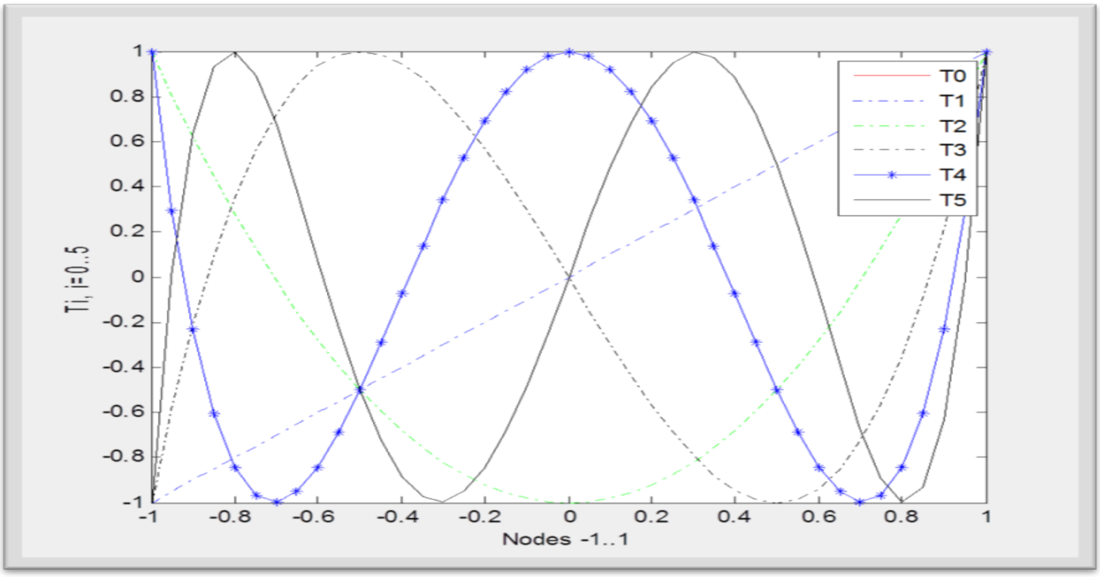

Figure 1: The first few Chebyshev polynomials of the first kind for N=0,1,2,3,4,5.

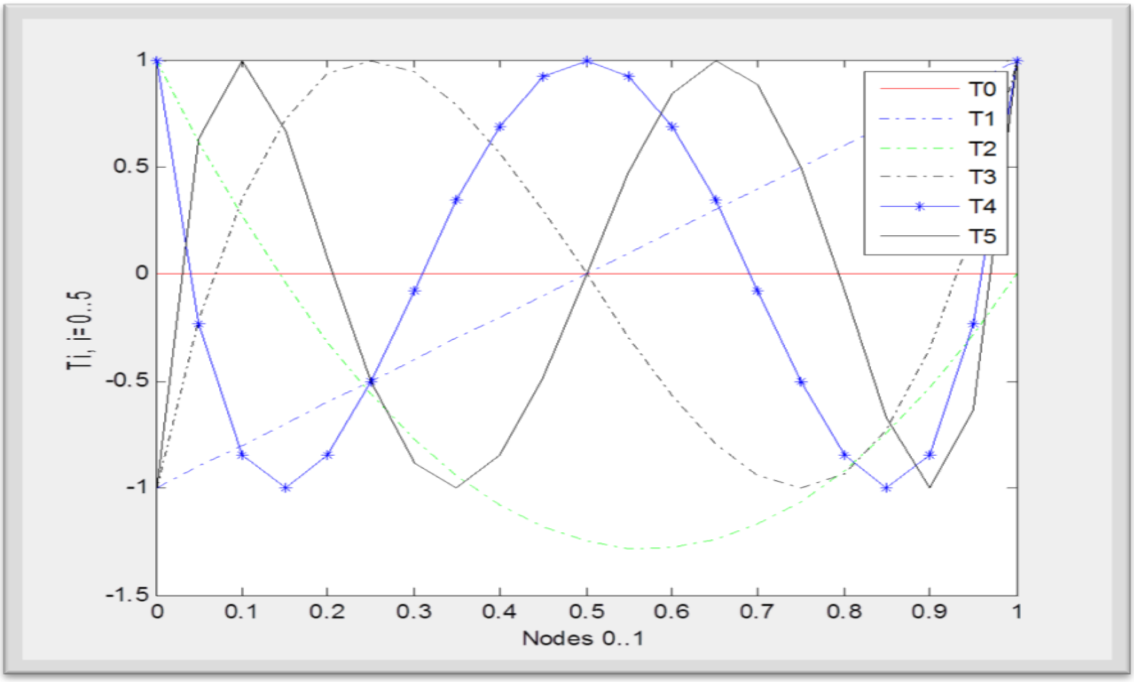

Figure 2: The first few Shifted Chebyshev polynomials of the first kind for N=0,1,2,3,4,5.

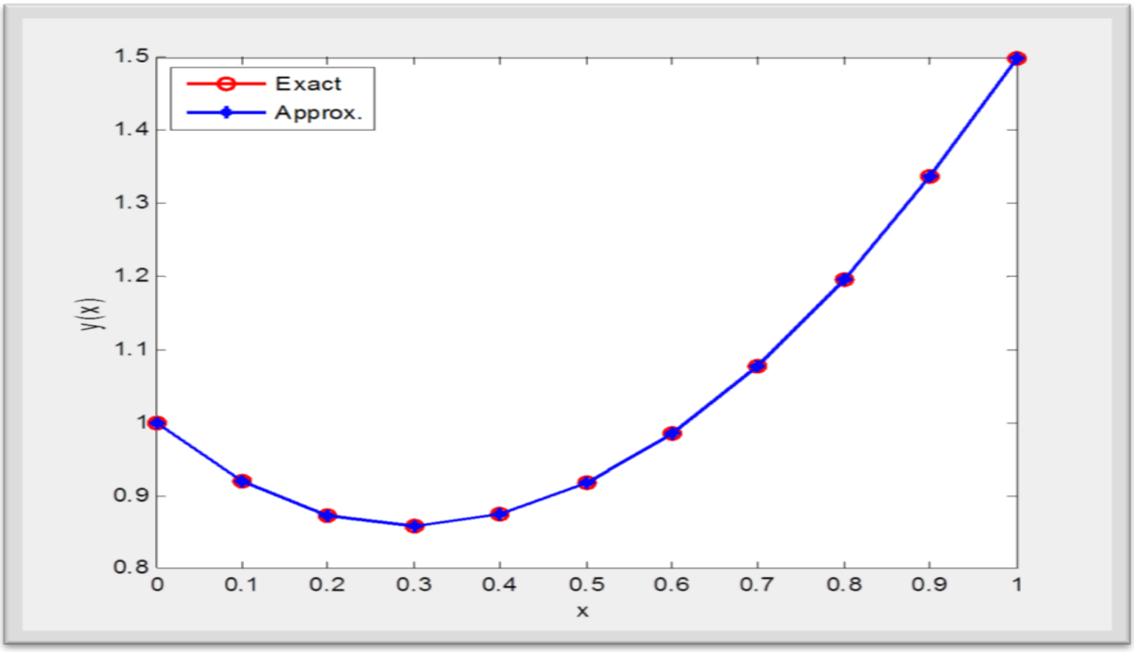

Figure 3: A Comparison between the exact and the approximate solution using expansion method of Chebyshev polynomials of the first kind of application(1) for N=10.

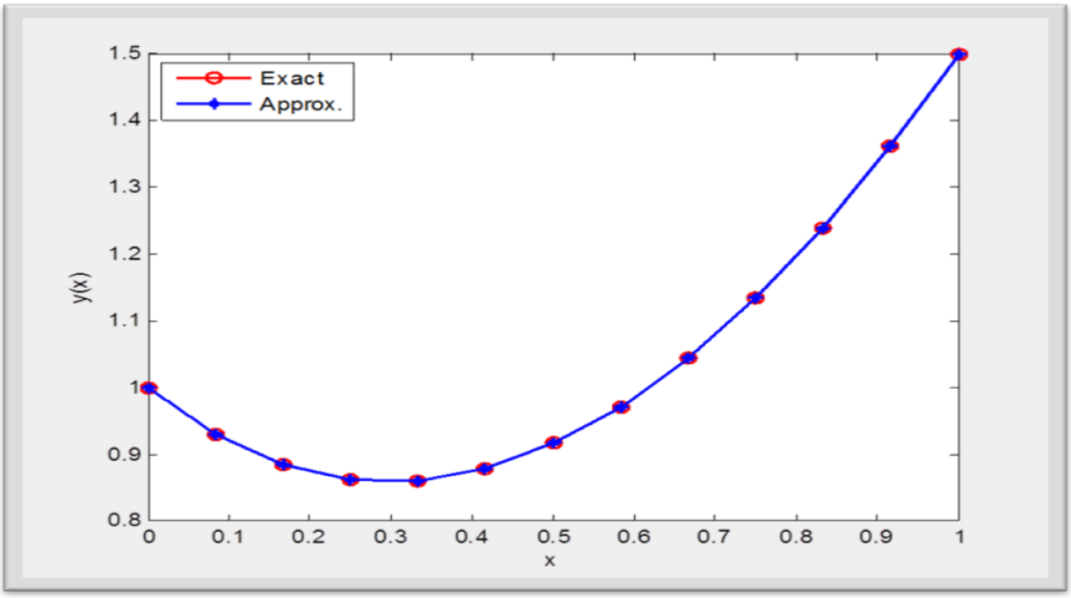

Figure 4: A Comparison between the exact and the approximate solution using expansion method of Chebyshev polynomials of the first kind of application(1) for N=12.

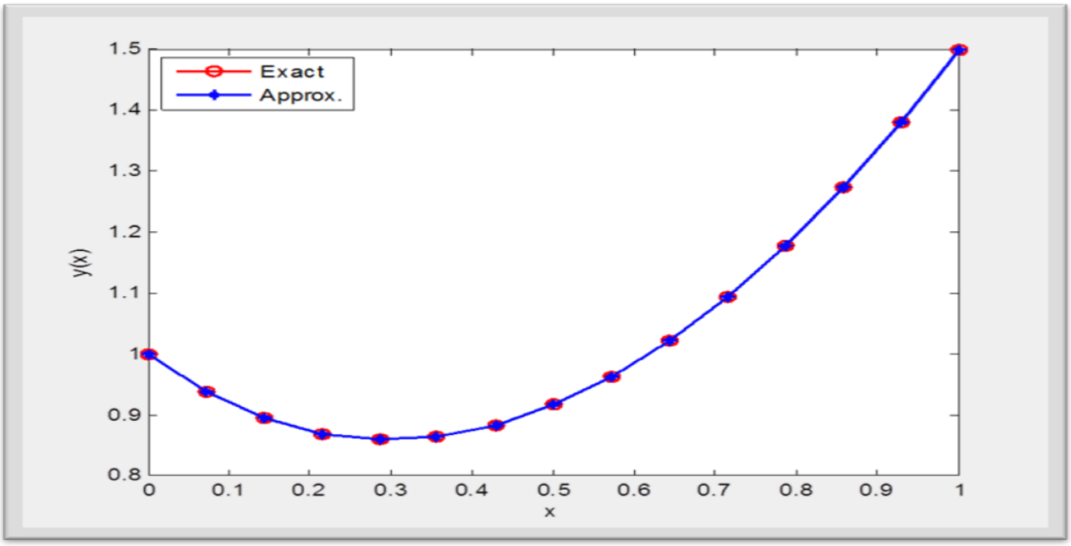

Figure 5: A Comparison between the exact and the approximate solution using expansion method of Chebyshev polynomials of the first kind of application (1) for N=14.

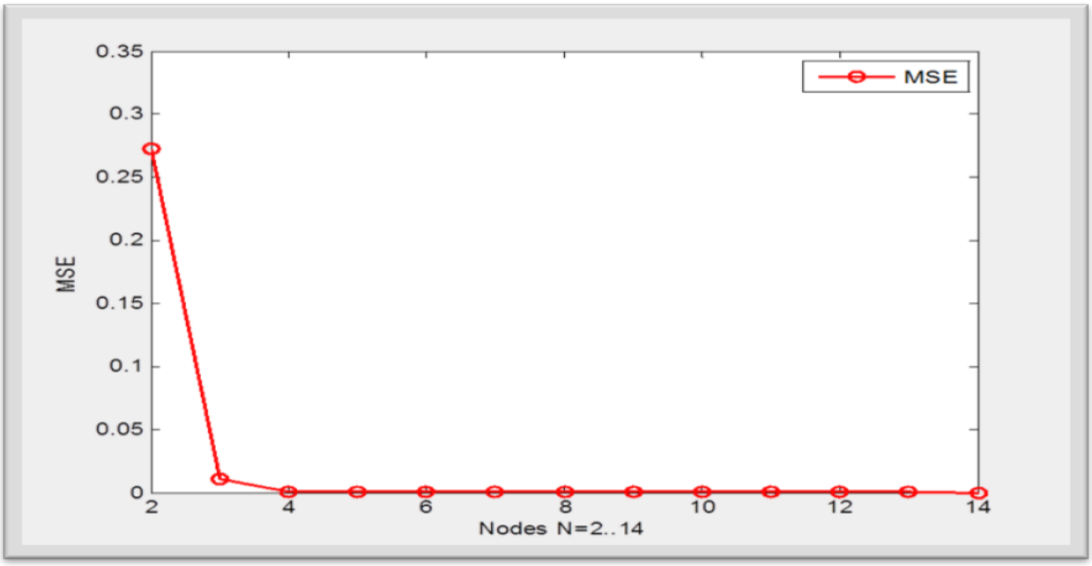

Figure 6: A Comparison between the Shifted Chebyshev nodes and the MSE of application (1) when N=2….14.

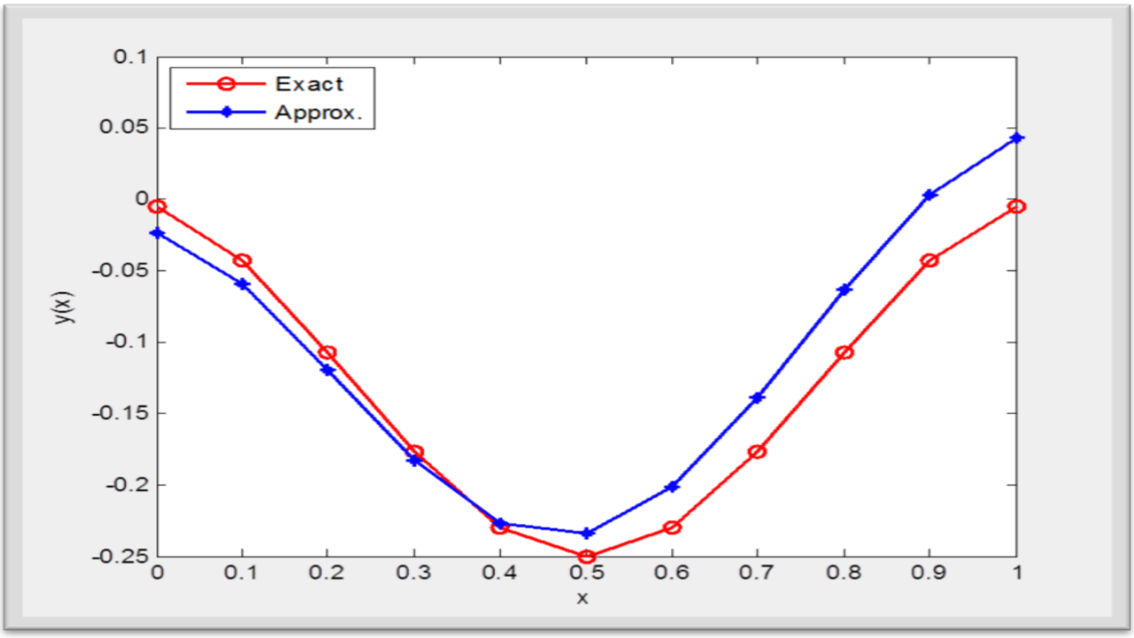

Figure 7: A Comparison between the exact and the approximate solution using expansion method of Chebyshev polynomials of application (2), for N=10.

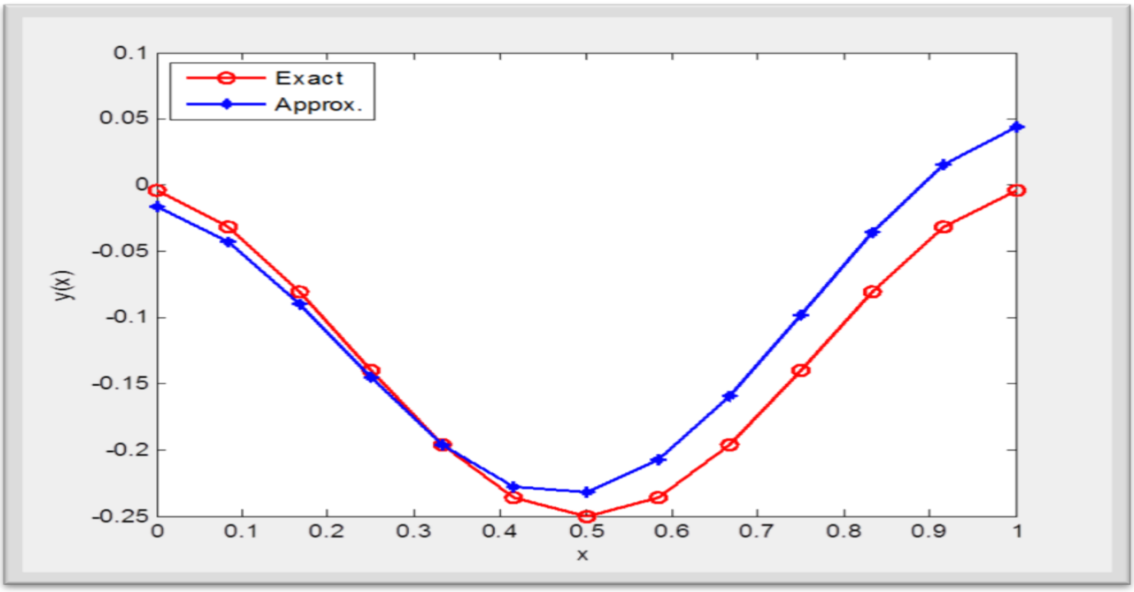

Figure 8:A Comparison between the exact and the approximate solution using expansion method of Chebyshev polynomials of application (2) for N=12.

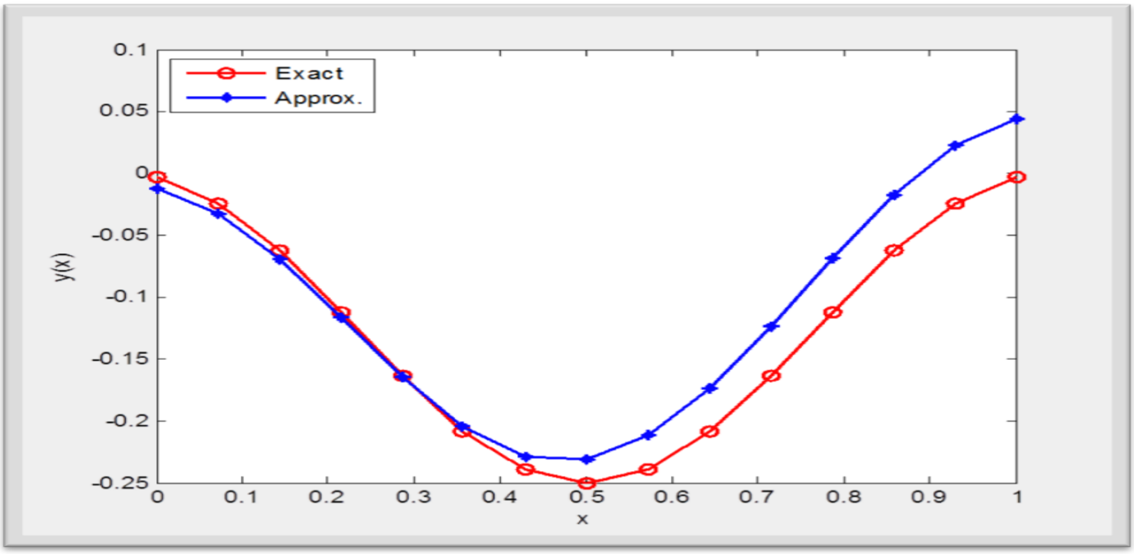

Figure 9: A Comparison between the exact and the approximate solution using expansion method of Chebyshev polynomials of application (2) for N=14.

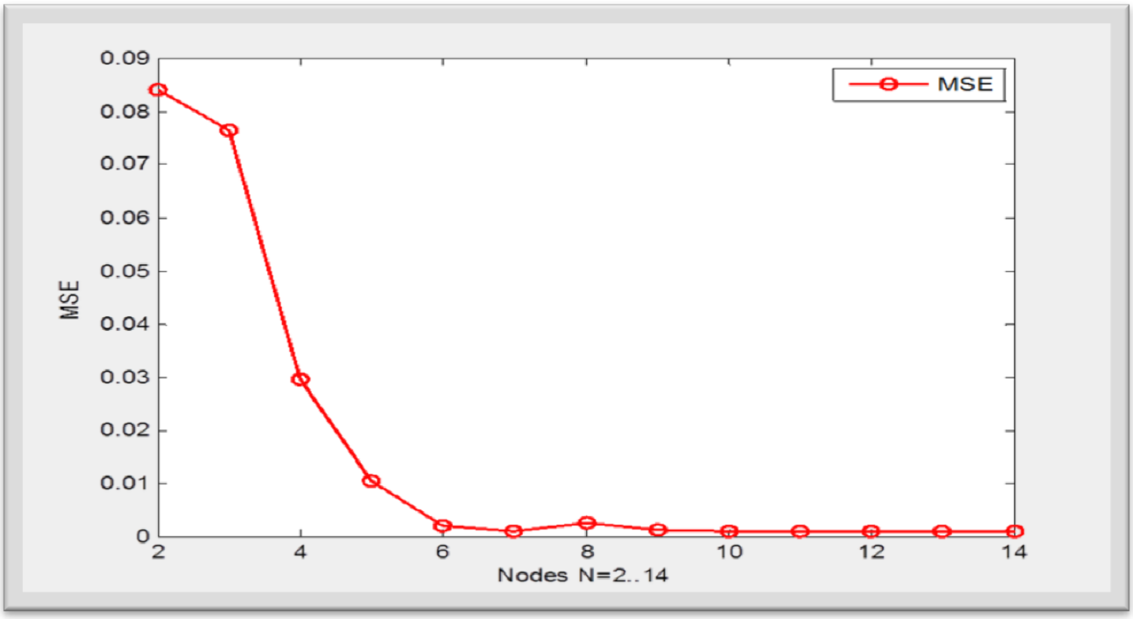

Figure 10: A Comparison between the Shifted Chebyshev nodes and the MSE of application (2) when N=2….14.